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Abstract. We recall the properties of the intelligent (and quasi-intelligent) spin states introduced by Aragone *et al* (1974 *J. Phys. A: Math. Gen.* **7** L149, 1976 *J. Math. Phys.* **17** 1963). We use these states to construct families of coherent wavepackets (WP) on the sphere and we sketch the time evolution of these WP for a rigid-body molecule.

1. Introduction

The eigenstates of the square of the angular momentum operator L^2 which are also eigenstates of $L_x + i\eta L_y$, where η is a real parameter, have been called *intelligent spin states* [1] and have been the subject of intensive analytical studies [2, 3]. However, at least to our knowledge, one does not find in the literature any discussion of their geometrical interpretation and of their utility in the construction of angular momentum coherent states. On completion of a recent paper [4] on the time evolution of coherent states built from a subclass of those states, we came to the conclusion that such a discussion still had to be presented.

In this short paper we first show in section 2 that the parameter η, enables one to define squeezed states on the sphere i.e. states for which the uncertainties ΔL_x^2 and ΔL_y^2 can be varied at will. It is, however, for real values of η that the product of the uncertainties has a minimum value. In section 3 we show that the states can be classified into three categories, a result already found in [3], and that the well known Radcliffe's states belong to one of these categories. In section 4 we study the angular localization and the partial wave expansion of a particular class generated from an exponential coherent state on the sphere. This state was introduced by us in [4]. Finally, in section 5 we study the time evolution of this particular set of states in the same fashion as [4] i.e. assuming that the wavepackets (WP) evolve with the Hamiltonian of a rigid body with moment of inertia J:

$$H = \frac{\hbar^2}{2J} L^2. \tag{1}$$

In particular, we show that the scenario of fractional revivals found in [8] is well exhibited by the family of WP studied in section 4.

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2. Squeezed states on the sphere

Let us enumerate a few general properties of the states $|w, \eta\rangle$ which are eigenstates of $L_x + i\eta L_y$ with a complex value of η. In this first part there is no need to assume that they are eigenstates of L^2. These states are the normalized states which obey the equation:

$$(L_x + i\eta L_y)|w, \eta\rangle = w|w, \eta\rangle. \tag{2}$$

It is a simple exercise, as already discussed in [5, 6] to prove equations (3)–(7). First of all

$$|\eta|^2 = \frac{\Delta L_x^2}{\Delta L_y^2} \tag{3}$$

i.e. $|\eta|$ can be called the squeezing parameter (with the definitions $\Delta L_i^2 = \langle L_i^2 \rangle - \langle L_i \rangle^2, i = x, y$). The phase α of η determines the ratio of the average value of the anticommutator of L_x with L_y to the average of their commutator since

$$\tan \alpha = \frac{\langle \{L_x, L_y\} \rangle - \langle L_x \rangle \langle L_y \rangle}{\langle L_z \rangle}. \tag{4}$$

Finally, the product of the uncertainties is given by:

$$\Delta L_x^2 \Delta L_y^2 = \frac{1}{4}[\langle L_z \rangle^2 + |\langle \{L_x, L_y\} \rangle - \langle L_x \rangle \langle L_y \rangle|^2] = \frac{1}{4}\frac{\langle L_z \rangle^2}{\cos^2 \alpha}. \tag{5}$$

The average values of L_x and L_y are both fixed by the parameter η and the eigenvalue w by:

$$\langle L_x \rangle = \frac{\eta w^* + w \eta^*}{\eta + \eta^*} \qquad \langle L_y \rangle = \frac{1}{i} \frac{w - w^*}{\eta + \eta^*}. \tag{6}$$

The eigenstates corresponding to the real parameter η satisfy the important minimum uncertainty relation from which [1] was initiated:

$$\Delta L_x^2 \Delta L_y^2 = \tfrac{1}{4}\langle L_z \rangle^2. \tag{7}$$

3. Classification of the squeezed states

The eigenvalue, w, can be obtained very simply in the basis of eigenstates of L^2 with the eigenvalue $l(l + 1)$ if one uses the observation [3] that, within a constant factor $\sqrt{1 - \eta^2}$, the operator $L_x + i\eta L_y$ is one of the three generators of an $SU(2)$ algebra. The set of operators satisfying this algebra is defined as

$$\mathcal{L}_3 = \frac{L_x + i\eta L_y}{\sqrt{1 - \eta^2}} \qquad \mathcal{L}_\pm = \pm \left(\frac{\eta L_x + i L_y}{\sqrt{1 - \eta^2}} \right) - L_z. \tag{8}$$

Therefore, there are $2l+1$ solutions to equation (2); instead of w one simply uses the eigenvalue of $\mathcal{L}_3$ with $k = -l, \ldots, +l$ and the formula

$$w = k\sqrt{1 - \eta^2}. \tag{9}$$

Let us denote by $|l, k, \eta\rangle$ the eigenstates solutions of equation (2) in the basis where L^2 is diagonal. (Note that k is the eigenvalue of $\mathcal{L}_3$ and not of L_z.) Using (9) one obtains two expressions for the average of L_x and L_y in the particular case of a real value of η:

$$\langle L_x \rangle = k\sqrt{1 - \eta^2} \qquad \langle L_y \rangle = 0 \qquad \text{if} \quad |\eta| < 1 \tag{10}$$

$$\langle L_x \rangle = 0 \qquad \langle L_y \rangle = k\sqrt{\eta^2 - 1} \qquad \text{if} \quad |\eta| > 1. \tag{11}$$

The case with $|\eta| = 1$ is obviously singular but there is a unique well known solution for which $w = 0$. One then has

$$(L_x \pm iL_y)|l, k = \pm l, \eta = \pm 1\rangle = 0 \tag{12}$$

and $|l, \pm l, \eta = \pm 1\rangle$ coincide with the eigenstates of L_z with eigenvalues $\pm l$. If η takes a complex value, $\langle L_x \rangle$ and $\langle L_y \rangle$ are both nonzero but are both proportional to k. The intelligent spin states are the solutions of equation (2) with average values given by (10) or (11) and which moreover satisfy equation (7). The *generalized intelligent states* or the *quasi-intelligent spin states* which were respectively defined in [1, 3] are the solutions of equation (2) with a complex value of η for which both averages are nonzero and to which equation (5) must be applied with a nonzero value of α. In both cases it is sufficient to consider the interval $|\eta| < 1$ and $\alpha \in [0, \pi/2]$.

Let us now classify those states according to k as follows:

 (i) The states with $k = 0$ for which $\langle L_x \rangle$ and $\langle L_y \rangle$ are zero and only $\langle L_z \rangle$ is nonzero.
 (ii) The states with $k = \pm l$ which are very particular as we show below.
(iii) The states with intermediate values of k.

In order to justify this classification we have to express $\langle L_z \rangle$ as

$$\langle L_z \rangle = \eta \langle \mathcal{L}_3 \rangle + i\sqrt{1 - \eta^2} \langle L_y \rangle - \langle \mathcal{L}_+ \rangle. \tag{13}$$

Except for the case $k = \pm l$ the average of the operator $L_x + i\eta L_y$ defined by equation (8) is nonzero. This average can be calculated from the works of [1, 3]. It is generally the ratio between two polynomials of the parameter η, the degree of which increases with l. Therefore, the angle between the vector $\langle \vec{L} \rangle$ and the Oz-axis is not a simple function of η for general values of k and l and there is no simple geometrical meaning of this angle. For $k = \pm l$ such an interpretation can indeed be found. Let us define η in terms of two angles θ_0 and ϕ_0 by

$$\eta = \frac{\tan \phi_0 + i \cos \theta_0}{\cos \theta_0 \tan \phi_0 + i}. \tag{14}$$

With this value of η the equation

$$(L_x + i\eta L_y)|l, l, \eta\rangle = l\sqrt{1 - \eta^2}|l, l, \eta\rangle \tag{15}$$

can be written simply as

$$(\vec{L} \cdot \vec{l})|l, l, \eta\rangle = l|l, l, \eta\rangle \tag{16}$$

where the vector $\vec{l}$ is the unit vector in the direction of $\langle \vec{L} \rangle$ which is such that

$$\langle L_z \rangle = l \sin \theta_0 \cos \phi_0 \qquad \langle L_y \rangle = l \sin \theta_0 \sin \phi_0 \qquad \langle L_z \rangle = l \cos \theta_0. \tag{17}$$

Equation (17) expresses the fact that $|l, l, \eta\rangle$ are simply spherical harmonics, Y_l^l, in a system of axis where Oz is along the vector l. These states are generally called Radcliffe's states [7]. Let $\vec{u}, \vec{v}, \vec{l}$ be three unit vectors forming an orthogonal direct system. One has also

$$(\vec{L} \cdot (\vec{u} + i\vec{v}))|l, l, \eta\rangle = 0. \tag{18}$$

In this rotated system of coordinates the parameter η is equal to one†!

For $k \neq \pm l$ the transformation which enables one to build up the intelligent or quasi-intelligent spin states from the usual spherical harmonics is not a rotation. This remark was already formulated long ago in [3] at the same time as the previous remark on the equivalence of Radcliffe's states with the states with $k = \pm l$.

† One also finds other cases where the states coincide with spherical harmonics with an axis of quantization different from Oz: it is the Ox-axis for $\eta = 0$ and for $\eta = i|\eta|$ this axis has $\theta = \pi/2$ and $\sin \phi = -|\eta|/\sqrt{1 - \eta^2}$.

4. Study of a family of exponential WPs

Let us now discuss the properties of WP built from intelligent or quasi-intelligent spin states with the same value of w, i.e. of k, and containing many different values of l. The consideration of such admixtures indeed enables us to modify, at will, the angular spread keeping either (5) or (7), with the allowance, provided by equation (3), that η is an adjustable squeezing parameter. These WP can be expressed either in the basis $|l, k, \eta\rangle$ with coefficients $c_l(k, \eta)$ as

$$|\Psi_{\eta k}\rangle = \sum_l c_l(k, \eta)|l, k, \eta\rangle \tag{19}$$

or in the basis of ordinary spherical harmonics with coefficients $b_{lm}(k, \eta)$ as

$$\Psi_{\eta k}(\theta, \phi) = \sum_{lm} b_{lm}(k, \eta) Y_m^l(\theta, \phi). \tag{20}$$

From now on we will discuss the properties of families of states generated from a *parent* state previously investigated by us in [4] and and called the *exponential coherent state*. For a real value of η this state depends on a single parameter N, and a function of θ and ϕ called v. Its expression is:

$$\Psi_{\eta 0}(\theta, \phi) = \sqrt{\frac{N}{2\pi \sinh 2N}} e^{Nv} = \sqrt{\frac{N}{2\pi \sinh 2N}} e^{N \sin \theta(\cos \phi + i\eta \sin \phi)}. \tag{21}$$

The probability density which is associated depends only on N. It is maximum within a solid angle symmetric around the Ox-axis and the width of this solid angle is of the order $4\pi/(4N + 1)$. Also, the average $\langle L_z \rangle$ obeys the following formula

$$\langle L_z \rangle = \eta(N \coth(2N) - \tfrac{1}{2}) \xrightarrow{N \gg 1} \eta(N - \tfrac{1}{2}). \tag{22}$$

We have shown, in [4], how to obtain this WP from a three-dimensional harmonic oscillator coherent state. As stated in [4], all the WP have $k = 0$. Our purpose is now to construct for real or complex η families of WP having increasing values of k using (21) for the parent state. We would like to study the angular spread as a function of k and coherence properties of the following family:

$$\Psi_{\eta 0} \qquad \mathcal{L}_+ \Psi_{\eta 0} \qquad \mathcal{L}_+^2 \Psi_{\eta 0}, \ldots, \mathcal{L}_+^k \Psi_{\eta 0}. \tag{23}$$

These states are all solutions of equation (2) with respective eigenvalues $0, \sqrt{1 - \eta^2}, 2\sqrt{1 - \eta^2}$, $\ldots, k\sqrt{1 - \eta^2}, \ldots$. If η is real they all satisfy equation (7) while equation (5) is satisfied if η is complex. Applying $\mathcal{L}_+$ to the argument v of (21), one obtains a function v_+ defined by

$$v_+ = \mathcal{L}_+ v = \frac{1}{2}(\cos \theta - \eta)\sqrt{\frac{1 + \eta}{1 - \eta}} - \frac{1}{2}(\cos \theta + \eta)\sqrt{\frac{1 - \eta}{1 + \eta}} + (\eta \cos \phi + i \sin \phi) \sin \theta. \tag{24}$$

But a second application leads to the following property:

$$\mathcal{L}_+ v_+ = \mathcal{L}_+^2 v = 0. \tag{25}$$

The set defined by (23) can then be simply expressed in terms of $\Psi_{\eta,0}$ and of powers of v_+ as

$$\Psi_{\eta 0} \qquad v_+ \Psi_{\eta 0} \qquad v_+^2 \Psi_{\eta 0}, \ldots, v_+^k \Psi_{\eta 0}. \tag{26}$$

In this manner one sees that the Gaussian will be dominant for any k in such a way that the WP will be highly concentrated on the sphere even for high k. The action of the operator $\mathcal{L}_+^k$ is, however, very different if one analyses the decomposition of WP into partial waves according to the expansion (20). Indeed, it suppress all the partial waves with $l < k$, it moves the distribution towards the values with $m > l$ and also, by the change in the normalization, it increases the weight of the higher partial waves. However, if one uses not the basis of the
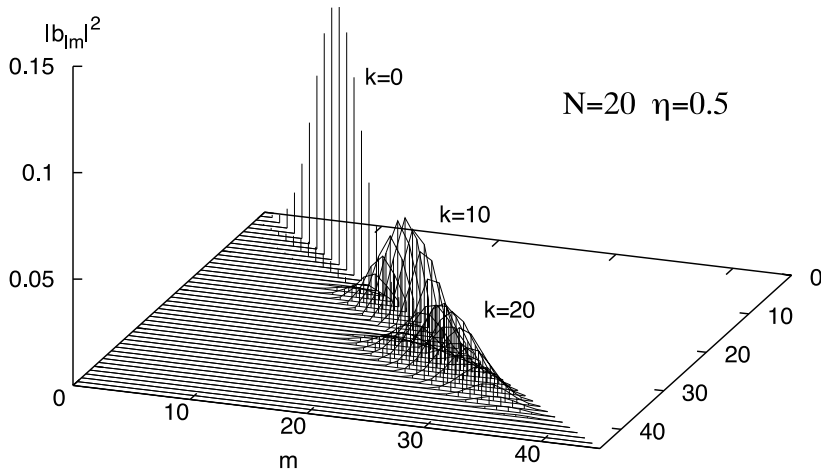
Figure 1. Distribution of weights $|b_{lm}|^2$ for the WP with $N = 20$, $\eta = 0.5$ and three values of k. The cases with $k = 10$ and 20 are plotted both in lines and impulses.

usual spherical harmonics (20) but the basis of the intelligent spin states one can draw benefit from the fact that we have in this basis the equation

$$\mathcal{L}_+|l, k, \eta\rangle = \sqrt{l(l + 1) - k(k + 1)}|l, k + 1, \eta\rangle \tag{27}$$

which shows that the relative phases of the states present in (19) are not affected. Therefore the states will keep their coherence property as a function of k.

To be complete, below we give the expression of $b_{lm}(0, \eta)$ from which a recurrence can easily be built to find the coefficients for $k > 0$. One found in [4] that for real values of η

$$b_{lm} = \sqrt{\frac{2N}{\sinh(2N)}} \sum_{l_1 l_2}(-1)^{l_2} \frac{(N(1 + \eta))^{l_1}(N(1 - \eta))^{l_2}}{\sqrt{(2l_1)!(2l_2)!}} \frac{\langle l_1 l_2 00|l0\rangle \langle l_1 l_2 l_1 - l_2|lm\rangle}{\sqrt{(2l + 1)}}. \tag{28}$$

For complex η the square root in front must be changed.

The distribution of the coefficients b_{lm} is presented in figure 1 for $\eta = 0.5$, $N = 0$ and $k = 0$, 10 and 20. A very large shift of the distribution of the bs towards the higher l and $m > l$ is clearly observed when k increases as well as an increase in the spread of the values of m.

Sections of the WP at their maximum are presented in figure 2 which shows the angular shape of the WP. Apart from a small movement towards smaller values of θ when k increases the WP is almost as strongly localized for $k = 20$ as it was for $k = 0$ for $\eta = 0.5$ and $\alpha = 0$, as well as for $\alpha \neq 0$.

Despite the strong angular concentration of the WP for $k = 20$, the calculation of $\langle L_z\rangle$ provides very large values for the product of the uncertainties. This average value is shown in figure 3 for $\eta = 0.5$ at the same time as $\langle L^2\rangle$.

5. Fractional revivals for the case of a rigid rotation

The time evolution of the WPs described above can now be studied assuming the Hamiltonian (1). This assumption was applied in [4] for the state (21), i.e. the state representing a rigid heteronuclear molecule and we have made an extensive study of its time evolution. The fractional waves that one obtains as time proceeds are obtained from the initial WP by multiplying the b_{lm} by $\exp(-il\omega_0 t_s)$ (see [4]) where ω_0 is the frequency of periodicity of
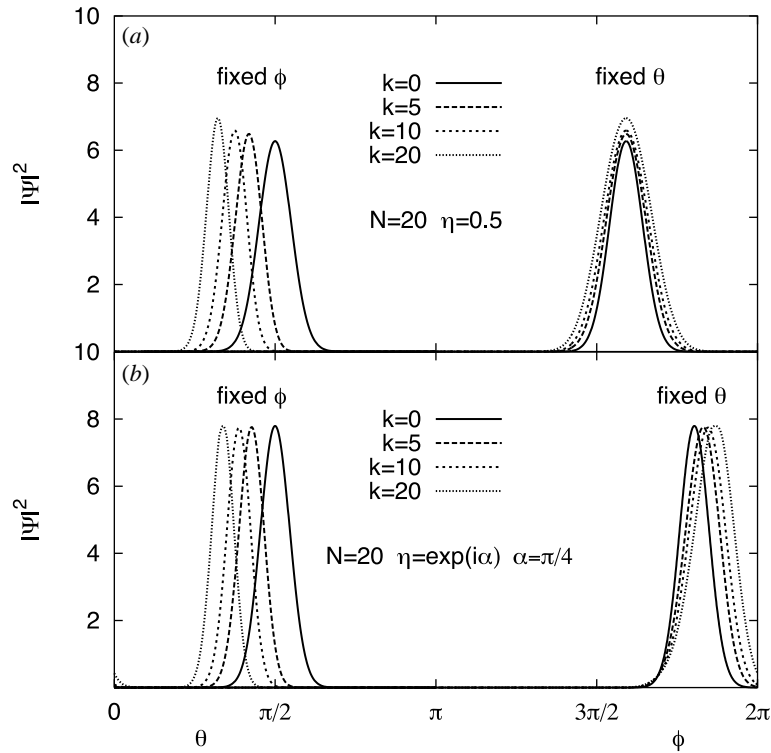
Figure 2. Sections, in θ and ϕ directions, through the WP maxima at time $t = 0$ for WP with $N = 20$, $\eta = 0.5$ (*a*), and those with $N = 20$, $\eta = \exp(i\alpha)$, $\alpha = \pi/4$ (*b*) and four k values. In (*a*) all sections in ϕ are shifted from $\phi = 0$–5 for more convenient presentation.
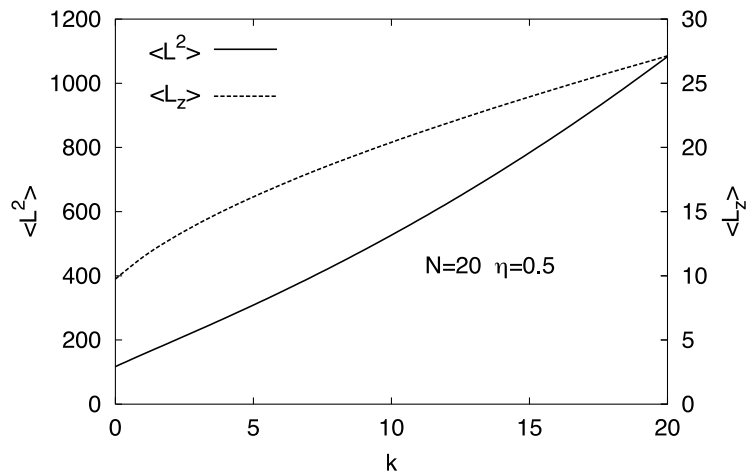

Figure 3. Expectation values of L^2 (left scale) and L_z (right scale) operators for WP with $N = 20$, $\eta = 0.5$ as functions of k.

the rigid rotor and where t_s is a fractional time. It is this change of t_s which allows one to obtain a rich variety of fractional waves on the sphere for $k = 0$. Depending on the value of
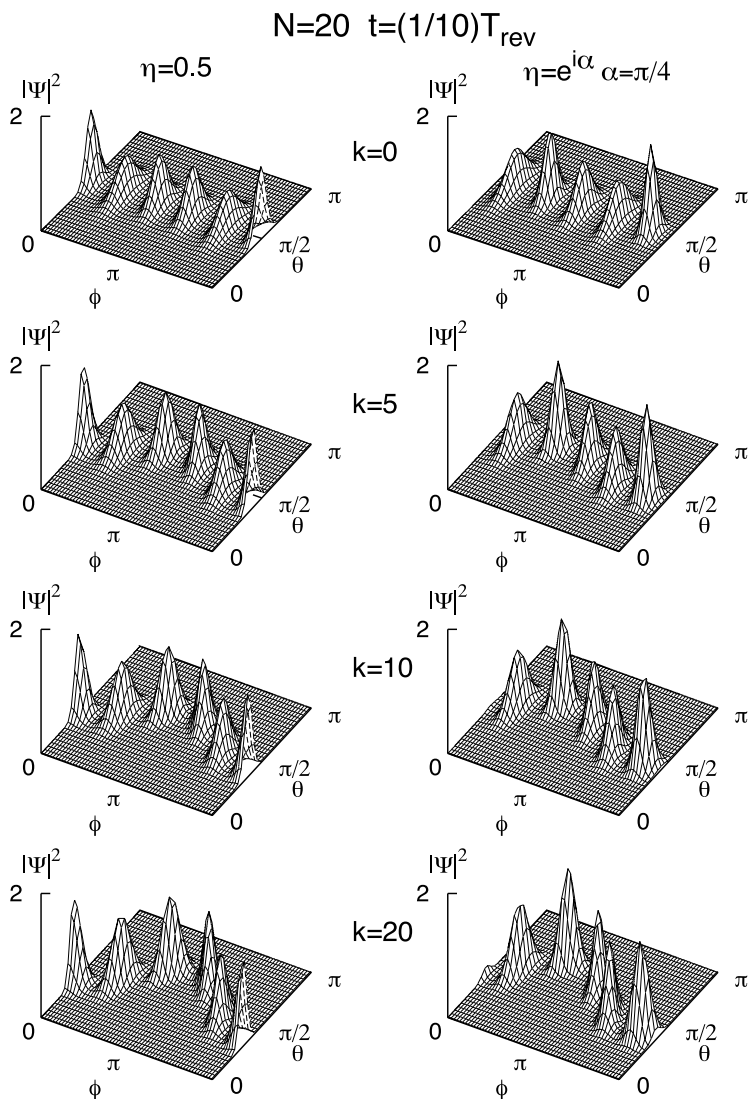
Figure 4. Shapes of WP with $N = 20$ at fractional revival time $t = (\frac{1}{10})T_{rev}$ for real $\eta = 0.5$ (left column), and $\eta = \exp(i\alpha)$, $\alpha = \pi/4$ (right column) and $k = 0, 5, 10, 20$ as functions of angular variables for a rigid molecule. Clones and mutants are clearly visible. Note that with increasing k the classical trajectory becomes more and more tilted with respect to the Oxy-plane.

η (only real values of η were considered in [4]) the WP exhibits a rich scenario of fractional revivals in the way described in [8], the origin of which being traced to the quantum mechanical spreading. The parameter η allows one to control the relative spread of each of the angular variables. For fractional time $(m/n)T_{rev}$, where T_{rev} is a common revival time, the WP is subdivided into a certain number of WP (n if n is odd, $n/2$ in the even case) the shape of which depend strongly on the value of η. For the case $\eta = \pm 1$ the fractional WP are clones of the initial WP. For different values their shape changes (we have called these WP *mutants*). If N is large enough the fractional WP are located around different directions on the sphere and do not spatially interfere much for low enough values of n ($n < 8$ for $N = 20$). We interpret

these properties as a manifestation of a robust virtue of coherence of the WP. In figure 4 the revivals of various WP having $k = 0, 5, 10$ and 20 are presented for $m/n = \frac{1}{10}$, such a time has been chosen as a typical example. There we should observe five WP according to [8]. These WP are always very well separated from each other, again there is no significant difference between the WP constructed from intelligent spin states and those built from quasi-intelligent ones.

6. Conclusions

The conclusion of this paper is that there exists numerous possibilities of the construction of angular coherent states using the properties of the intelligent spin states. For a system with a Hamiltonian quadratic in I [$I(I + 1)$ spectrum] these WP spread on the sphere but there is a well identified mechanism of fractional revivals that produce a set of well concentrated *mutants*. For rotators which are not quadratic the scenario is valid during a limited time. This is the case of nuclei for which we are making a parallel study [9].

Acknowledgment

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